1965

Vibration analysis of three dimensional piping systems with general topology

Baird, Winfield Scott, Jr.

Monterey, California: U.S. Naval Postgraduate School

http://hdl.handle.net/10945/13361

VIBRATION ANALYSIS OF THREE DIMENSIONAL PIPING SYSTEMS WITH GENERAL TOPOLOGY

WINFIELD SCOTT BAIRD

VIBRATION ANALYSIS OF

THREE DIMENSIONAL PIPING SYSTEMS

WITH GENERAL TOPOLOGY

* * * * *

Winfield Scott Baird, Jr.

VIBRATION ANALYSIS OF

THREE DIMENSIONAL PIPING SYSTEMS

WITH GENERAL TOPOLOGY

by

Winfield Scott Baird, Jr.
Lieutenant Commander, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
MECHANICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California

1 9 6 5

VIBRATION ANALYSIS OF THREE DIMENSIONAL PIPING SYSTEMS WITH GENERAL TOPOLOGY

by

Winfield Scott Baird, Jr.

This work is accepted as fulfilling

the thesis requirements for the degree of

MASTER OF SCIENCE

IN

MECHANICAL ENGINEERING

from the

United States Naval Postgraduate School

ABSTRACT

A theoretical analysis for the characteristic undamped natural frequencies and mode shapes of three dimensional piping systems with general topology is presented. The development makes use of matrix methodology including transfer matrices, dynamic stiffness matrices, and certain topological matrices derived from the theory of linear graphs. A distributed mass model is employed throughout.

TABLE OF CONTENTS

Chapter	Title	Page
I	Introduction	1
	1.1 General Remarks	1
	1.2 Scope of Work Presented	1
	1.3 Notation	2
II	Definition of Properties and Operations	4
	2.1 Problem Nomenclature & General Procedure	4
	2.2 State Vector	6
	2.3 Transfer Matrix	8
	2.3a Transfer Matrix Sign Convention	9
	2.4 Dynamic Stiffness Matrix	10
	2.4a Dynamic Stiffness Matrix Sign Convention	11
	2.5 Local Solid Geometry	13
	2.5a Node Rotation Matrix	18
	2.6 Global Solid Geometry	20
	2.7 Application of Graph Theory and Topological Matrices	24
	2.8 Boundary Conditions and the Frequency Determinant	27
	2.9 Mode Shapes	32
III	Discussion	34
	3.1 Network Analogies	34
	3.2 Practical Limitations	34
	Bibliography	36
	Appendix	
	A Distributed Mass Transfer Matrix for Circular Arcs	
	B Sub-system Topological Technique	43

CHAPTER I

Introduction

1.1 General Remarks

The dynamic behaviour of piping systems is of general interest in the design of such systems. The objective of this thesis is to present an analytical method of determining the characteristic frequencies and mode shapes of mechanical vibration of three dimensional piping systems possessing general topology.

Although no digital computer program is incorporated in this thesis, the matrix methods employed and the iteration solution suggested not only lend themselves to a computer solution, but they make a large capacity, high speed digital computer mandatory.

1.2 Scope of Work Presented

A method for determining, analytically, the undamped natural frequencies and mode shapes of three dimensional piping systems with general topology is presented herein. Previous investigators in this field have presented techniques which have been successfully employed for analysis of systems contained entirely in a plane and with "tree like" topology, that is, elastic branches have been permitted, but involved meshes or loops have been excluded. Also, previous mechanical models have not permitted the analysis of circular, elastic arcs with distributed mass. A method of obtaining the transfer matrix for a circular arc with distributed mass is presented in Appendix A. The details of the method of obtaining topological matrices which permit analysis of general topology is presented in Chapter 2, Section 2.7. Dynamic stiffness matrices are defined in Chapter 2, Section 2.4. The details of the necessary geometric analysis are presented in Chapter 2, Section 2.5 and 2.6.

1.3 Notation.

[ ]	matrix
[SV]	state vector
sv_i	i th element of a state vector
$[Z]_{ab}$	transfer matrix from node a to node b
$[D]_i$	deflection partition of a state vector
$[f]_i$	force partition of a state vector used with transfer matrix sign convention
$[Z_{11}]$,etc	six by six partition of a transfer matrix
$[S]_{a,b}$	dynamic stiffness matrix between node a and node b
$[S_{11}]$,etc	six by six partition of a dynamic stiffness matrix
$[F]_i$	force partition of a state vector, associated with dynamic stiffness matrix sign convention
$[D]_{a,b}$	deflection matrix of the terminal nodes of a primary path
$[F]_{a,b}$	force matrix of the terminal nodes of a primary path
X,Y,Z	system coordinates of a node
L	vector length
α, β, γ	direction cosines
$\bar{e}$	unit vector
θ	central included angle
$\bar{\rho}$	radius of curvature
$\overline{AO}$	vector from point A to point O
[G]	node rotation matrix
[TLG]	state vector transformation matrix from a local reference system to the global reference system

[TGL]	state vector transformation matrix from the global reference system to a local reference system
[SGL]	primary path deflection matrix transformation matrix from global reference system to a local reference system
[SLG]	primary path force matrix transformation matrix from local coordinates to global coordinates
G	network
[A]	node incidence matrix
[SP]	primitive stiffness matrix
[SN]	node stiffness matrix
$\lvert\overline{SN}\rvert$	frequency determinant
[DM]	mode shape matrix
[FN]	node force matrix
[DN]	node deflection matrix
λ	eigenvalue of an array of coefficients of differential equations

CHAPTER II

Definition of Properties and Operations

2.1 Problem Nomenclature and General Procedure

The solution for the characteristic frequencies of a piping system with general topology requires a form of network analysis. In the analysis which follows, a given piece of pipe will be referred to as an element. The ends of an element will be incident on nodes. If only two elements are incident on a node, that node will be referred to as a trivial node. When more than two elements are incident on a node, that node will be referred to as a primary node. When an element terminates at a foundation or some other boundary of the system, that node, upon which only one element is incident, will be referred to as a boundary node. Those elements connecting two non-trival nodes together will form a primary path when only trivial nodes are in the path connecting the two non-trivial nodes.

The necessary solid geometry analysis utilizes conventional vector notation and involves many different frames of reference. Right handed triads will be developed which will be associated with each end of each element and will generally be referred to as local triads or local reference systems. Such a triad will be so oriented that the x coordinate will be tangent to the centroidal axis of the element involved. One triad will be designated a global triad or global reference system and the node at which the global reference system is located will be designated the global origin.

The general procedure is as follows. Beginning at a primary or boundary node and proceeding along a primary path, certain geometrical properties will be developed for the first element encountered which will permit computing a transfer matrix. The next element will then be analyzed for its geometrical properties, its transfer matrix will be computed, a node rotation matrix, which is a function of the two triads associated with the ends of the elements incident at the trivial node involved, will be computed, and the product of these three matrices will be formed so as to eliminate the trivial node and leave a single transfer matrix for the first two elements. The next element is then analyzed in a similar fashion, a new transfer matrix is formed, and so on until a single transfer matrix has been computed for the entire primary path. This transfer matrix will then be transformed into a dynamic stiffness matrix which will contain the characteristic dynamic properties of the primary path analyzed. The dynamic stiffness matrix will then be transformed to the global reference system. This procedure will be repeated for all primary paths.

Once dynamic stiffness matrices have been calculated for each primary path, they will all be assembled into a single diagonal matrix known as the primitive stiffness matrix. A topological matrix known as a node incidence matrix will then be formed. By premultiplying by the transpose of the node incidence matrix and postmultiplying by the node incidence matrix itself, the primitive stiffness matrix may be transformed into a new matrix called the node stiffness matrix. This matrix will relate the forces at the boundary nodes to the deflections at the non-trivial nodes of the network. By applying a qualitative knowledge of the constraints at the boundary nodes, a determinant known as the frequency determinant may be obtained from the node stiffness matrix. The frequency determinant will vanish for the characteristic frequencies of the network. Essentially, by a process of assuming values of frequency and determining the corresponding values of the frequency determinant, its zeros may be located and refined.

2.2 State Vector

The state vector at a given node, a, of an elastic system is the collection or group of quantities whose sense and magnitude completely describe the instantaneous displacement, both rectilinear and angular, of that point from its quiescent position, as well as the corresponding rectilinear and angular forces in the member at the same point at the same instant, that is, those forces which, if a cutting plane were passed through the point at the instant of interest, would have to be applied to each cut face to prevent relative motion between them. For the purposes of this discussion, the state vector will be considered to be a column matrix.

Although the systems to be studied are three dimensional, the individual elements which make up the systems will be planar, that is, only straight elements or circular arcs are permitted. For such planar elements, it is possible to consider an in-plane and an out-of-plane state vector. Each of these state vectors would have six elements, three pertaining to force and three to deflections. Although this notion has some utility when discussing systems contained entirely in a plane or when deriving the distributed mass transfer matrices (to be discussed later) of the individual planar elements, it is useless for describing a three dimensional system. Therefore, all future references to a state vector, symbolized by [SV], will be the complete three dimensional state vector containing twelve elements, six of which describe displacement, and the other six describing force. These elements are organized and defined as follows:

$$[SV] = \begin{bmatrix} \text{sv 1, displacement in an x direction} \\ \text{sv 2, displacement in a y direction} \\ \text{sv 3, displacement in a z direction} \\ \text{sv 4, rotation about an x direction} \\ \text{sv 5, rotation about a y direction} \\ \text{sv 6, rotation about a z direction} \\ \text{sv 7, force in an x direction} \\ \text{sv 8, force in a y direction} \\ \text{sv 9, force in a z direction} \\ \text{sv 10, moment about an x direction} \\ \text{sv 11, moment about a y direction} \\ \text{sv 12, moment about a z direction} \end{bmatrix} \qquad 2\text{-}1$$

A local x y z coordinate system and the elements of a state vector are arranged as in Fig. 2.1-1.

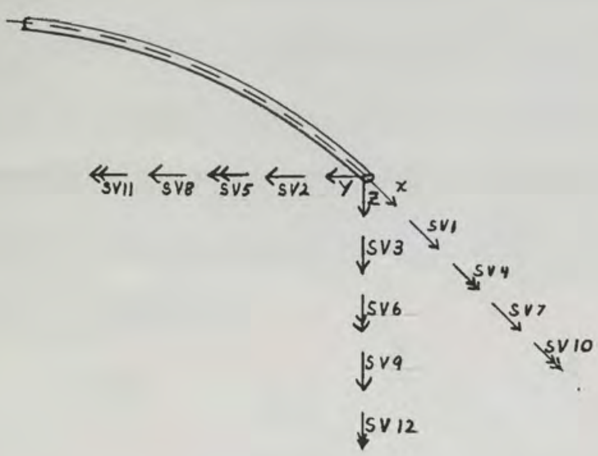


Fig. 2.1-1.

The x axis of a local coordinate system is tangent to the centroidal axis. The y axis is normal to the x axis and directed in a convenient manner. The z axis is normal to both the x and y axis so as to form a right handed triad. As an example, the y axis of a curved element may be fixed in the plane of the curve and directed towards the center of curvature. The z axis would then be normal to the plane of the curve. A more complete discussion of coordinates is contained in a general discussion of solid geometry which follows.

2.3 Transfer Matrix

The derivation of transfer matrices is treated in great detail in Reference 2 and the method employed to develop the transfer matrix for a circular arc with distributed mass is discussed in detail in Appendix A. Accordingly, only the briefest discussion will be given here, largely for the purpose of refreshing the reader's prior knowledge of this subject. For the purposes of this discussion, a transfer matrix is defined to be a twelve by twelve matrix which relates a state vector at one node and with a given orientation to another state vector at another node in a common primary path and generally having a different orientation from the first. Let [Z] represent such a transfer matrix. This relationship is described by the following matrix equation.

$$[SV]_b = [Z]_{ab} \cdot [SV]_a \qquad 2\text{-}2$$

In this equation, $[SV]_a$ represents a column state vector at node a, $[SV]_b$ represents another column state vector at node b, and $[Z]_{ab}$ represents the transfer matrix between node a and node b and is a frequency dependent property of the element or elements connecting node a and node b.

A dot written between two matrices indicates nothing more than ordinary matrix multiplication which could, of course, be indicated by writing the matrices in close juxtaposition. Here the dot is used to avoid confusion which might result from typewritten typography.

Consider the following system containing two elements designated ab and bc whose transfer matrices are known as $[Z]_{ab}$ and $[Z]_{bc}$.

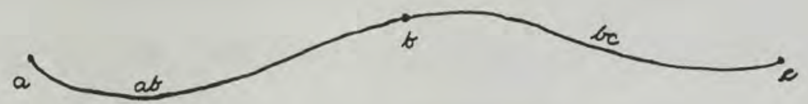


Fig. 2.2-1.

Given the state vector at node a, $[SV]_a$, the state vector at node b, $[SV]_b$, is given by eq. 2-2.

Furthermore, $[SV]_c$ is given as follows:

$$[SV]_c = [Z]_{bc} \cdot [SV]_b \qquad 2\text{-}3$$

It is apparent that the intermediate state vector at node b may be eleminated and the following equation written:

$$[SV]_c = [Z]_{bc} \cdot [Z]_{ab} \cdot [SV]_a \qquad 2\text{-}4$$

or

$$[SV]_c = [Z]_{ac} \cdot [SV]_a \qquad 2\text{-}5$$

where

$$[Z]_{ac} = [Z]_{bc} \cdot [Z]_{ab} \qquad 2\text{-}6$$

2.3a Transfer Matrix Sign Convention

A sign convention, unique to transfer matrix operation, must be established. The force elements of $[SV]_a$ are considered to be applied to element ab at node a by node a. They are positive when their sense is in the positive x, y, or z direction of the triad associated with element ab and located at node a. The angular forces, or moments, are positive according to a right hand rule about the x, y, or z directions. The force elements of $[SV]_b$, as computed by eq. 2-2, are those forces applied by element ab to node b and have sense and orientation according to the triad associated with element ab and located at node b. The force elements of $[SV]_b$, as given in eq. 2-3, are considered to be applied to element bc at node b by node b. Their sense and orientation are specified according to the triad located at node b and associated with element bc. In a like manner, the force elements of $[SV]_c$ are applied to node c by element bc and have sense and orientation according to the triad associated with element bc at node c.

It is apparent that there will generally be two local reference systems at each trivial node and that they will generally not have the same orientation. This problem in geometry is handled by a rotation matrix which has been omitted from eqs. 2-4 and 2-6 in the interests of clarity, but is included in a discussion of geometry which follows. (cf. Sect. 2.8)

2.4 Dynamic Stiffness Matrix

A given state vector may be partitioned into two smaller column matrices, one containing the first six elements pertaining to deflection and referred to as a displacement vector, [D], and a second column matrix containing the last six elements pertaining to forces and referred to as a force vector, [f].

Consider two nodes, a and b, connected by a primary path. The path connecting nodes a and b need not be a unique one. If there exists more than one path, this discussion applies to any given one. Let the two state vectors, $[SV]_a$ and $[SV]_b$, be partitioned into the associated displacement and force vectors, $[D]_a$, $[f]_a$, $[D]_b$, and $[f]_b$. Let the appropriate transfer matrix, $[Z]_{ab}$, be partitioned into four six by six matrices as indicated below.

$$\begin{bmatrix} D \\ \hline f \end{bmatrix}_b = \left[\begin{array}{c|c} Z_{11} & Z_{12} \\ \hline Z_{21} & Z_{22} \end{array}\right]_{ab} \cdot \begin{bmatrix} D \\ \hline f \end{bmatrix}_a \qquad 2\text{-}7$$

Equation 2-7 may be expressed in an equivalent form as the following two equations:

$$[D]_b = [Z_{11}] \cdot [D]_a + [Z_{12}] \cdot [f]_a \qquad 2\text{-}7a$$

$$[f]_b = [Z_{21}] \cdot [D]_a + [Z_{22}] \cdot [f]_a \qquad 2\text{-}7b$$

Let the following six by six matrices be defined as indicated.

$$[S_{11}] = [Z_{12}]^{-1} \cdot [Z_{11}] \qquad 2\text{-}8$$

$$[S_{12}] = -[Z_{12}]^{-1}$$

$$[S_{21}] = [Z_{21}] - [Z_{22}] \cdot [Z_{12}]^{-1} \cdot [Z_{11}]$$

$$[S_{22}] = [Z_{22}] \cdot [Z_{12}]^{-1}$$

We may now write the following.

$$-[f]_a = [S_{11}] \cdot [D]_a + [S_{12}] \cdot [D]_b \qquad 2\text{-}9$$

$$[f]_b = [S_{21}] \cdot [D]_a + [S_{22}] \cdot [D]_b \qquad 2\text{-}10$$

2.4a Dynamic Stiffness Matrix Sign Convention

Equation 2-9 computes the negative of the forces at node a applied by node a to element ab. If it is clearly understood that the negatives of these forces are the positive forces applied by member ab to node a, the negative sign in equation 2-9 may be eliminated and both $[f]_a$ and $[f]_b$ are then considered to be forces applied by element ab to nodes a and b oriented according to triads located at nodes a and b. It is emphasized that this constitutes a unique sign convention to be associated with stiffness matrices versus transfer matrices. In the succeeding development, force column matrices will be indicated by capital letters instead of lower case letters when they are associated with the stiffness matrix sign convention.

The two force vectors may now be combined into a single twelve element column matrix containing the forces applied by the element ab to the nodes at a and b. In like fashion, the two displacement vectors may be combined. If the four six by six matrices are also combined to form a twelve by twelve matrix, the following equation may be written.

$$\begin{bmatrix} F_a \\ F_b \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \cdot \begin{bmatrix} D_a \\ D_b \end{bmatrix} \qquad \text{2-11}$$

which we write as

$$[F]_{a,b} = [S]_{a,b} \cdot [D]_{a,b} \qquad \text{2-11a}$$

The twelve by twelve matrix, [S], will be referred to as the dynamic stiffness matrix. The dynamic stiffness matrix, like the transfer matrix from which it was derived, is a function of frequency and the relative displacement and orientation of the state vectors at a and b and also depends on the element or elements connecting a to b by a given primary path. One can think of eq. 2-11a as relating the distortion of the member, $[D]_{a,b}$, to the forces, $[F]_{a,b}$, producing them.

2.5 Local Solid Geometry

Analysis of a three dimensional piping system requires, for each element, a knowledge of the length, radius of curvature (for circular arcs), relative position and relative orientation. Such information may be readily obtained if the positions of the ends of each element are specified in a convenient basic reference system of cartesion coordinates. Such coordinates will be referred to as system coordinates and might be measured from a corner of a room or any other convenient reference point. The three coordinates will be referred to as X, Y, and Z. It is convenient also to develop a number of local coordinate systems for use in calculating transfer matrices and for other purposes. It is the purpose of this Section to accomplish this development.

Consider a single straight element whose X, Y, and Z coordinates are known. Let one end be designated an "a" end and the other a "b" end. The "a" end may be considered to be the end with which a known state vector is associated. Let the system coordinates of the "a" end be designated X_a, Y_a, and Z_a. In a like manner, the "b" end coordinates will be X_b, Y_b, and Z_b. Three local direction numbers may be defined as follows.

$$X_{ab} = X_b - X_a \qquad 2\text{-}12$$

$$Y_{ab} = Y_b - Y_a \qquad 2\text{-}13$$

$$Z_{ab} = Z_b - Z_a \qquad 2\text{-}14$$

These three numbers are the direction numbers of a vector directed form "a" to "b" and of length, L, equal to that of the straight element. L is given by the following equation.

$$L = \sqrt{X_{ab}^2 + Y_{ab}^2 + Z_{ab}^2} \qquad 2\text{-}15$$

Three direction cosines α_x, β_x, and γ_x may be obtained in the following manner.

$$\alpha_x = \frac{X_{ab}}{L} \tag{2-16}$$

$$\beta_x = \frac{Y_{ab}}{L} \tag{2-17}$$

$$\gamma_x = \frac{Z_{ab}}{L} \tag{2-18}$$

These three direction cosines are the direction cosines of a unit vector tangent to the centroidal axis of the straight element. The direction of this unit vector, designated $\bar{e}_x$, is defined to be the x direction of a unit triad fixed in space at the "a" end of the element. For a straight element, the y direction of the unit triad is defined so as to be parallel to the XY plane of the system coordinates as well as being normal to the local x direction. Given a unit vector, $\bar{e}_y$, directed in such a manner, the following vector equations may be written:

$$|\bar{e}_y| = 1 \tag{2-19}$$

$$\bar{e}_x \cdot \bar{e}_y = 0 \tag{2-20}$$

where $\bar{e}_x = \alpha_x \bar{i} + \beta_x \bar{j} + \gamma_x \bar{k}$ and $\bar{e}_y = \alpha_y \bar{i} + \beta_y \bar{j}$

For $\bar{e}_y$ vector with a zero Z component, equations 2-20 and 2-22 may be written in scalar form as follows:

$$\alpha_y^2 + \beta_y^2 = 1 \tag{2-19a}$$

$$\alpha_x \times \alpha_y + \beta_x \times \beta_y = 0 \tag{2-20a}$$

These two equations define the magnitudes of the direction cosines of $\bar{e}_y$ as follows:

$$\alpha_y = \frac{1}{\sqrt{1 + \left(\frac{\alpha_x}{\beta_x}\right)^2}} \tag{2-21}$$

$$\beta_y = -\frac{\alpha_x}{\beta_x \sqrt{1 + \left|\frac{\alpha_x}{\beta_x}\right|^2}} \qquad 2\text{-}22$$

$$\gamma_y = 0 \qquad 2\text{-}23$$

These equations will be valid except when $\bar{e}_x$ has zero Y component. In such a case, the following equations will be used:

$$\alpha_y = 0 \qquad 2\text{-}21a$$

$$\beta_y = 1 \qquad 2\text{-}22a$$

$$\gamma_y = 0 \qquad 2\text{-}23a$$

The z direction of the local triad at "a" may now be defined by the following vector equation:

$$\bar{e}_z = \bar{e}_x \times \bar{e}_y \qquad 2\text{-}24$$

The local triad thus established for a straight element specifies the orientation of the two state vectors associated with the ends of the element.

The second of the two types of elements permitted in a three dimensional piping system is a circular arc. In general, a piping system would be fabricated so that the ends of a circular arc would be tangent to the elements that they are connected with. Consider such an arc with ends designated "a" and "b" and with the system coordinates of the two ends designated X_a, Y_a, Z_a, X_b, Y_b, and Z_b in the same manner as the preceding straight element. The element attached to the arc at the "a" end will have a triad associated with it whose orientation has been previously determined. The unit vector, $\bar{e}_x$, of this triad will be tangent to the arc at the "a" end. Let this vector now be designated $\bar{e}_t$. Let the vector $\bar{C}$ connect the end points "a" and "b". The direction numbers of $\bar{C}$, C_1, C_2, and C_3 are given by the following equations.

$$C_1 = X_b - X_a \tag{2-25}$$

$$C_2 = Y_b - Y_a \tag{2-26}$$

$$C_3 = Z_b - Z_a \tag{2-27}$$

The length of $\bar{C}$ is given as follows:

$$L = \sqrt{C_1^2 + C_2^2 + C_3^2} \tag{2-28}$$

The direction cosines of C are as follows:

$$\alpha_c = \frac{C_1}{L} \tag{2-29}$$

$$\beta_c = \frac{C_2}{L} \tag{2-30}$$

$$\gamma_c = \frac{C_3}{L} \tag{2-31}$$

These three direction cosines define a second unit vector $\bar{e}_c$ contained in the plane of the arc just as is $\bar{e}_t$. A unit vector, $\bar{e}_n$, normal to the plane of the curve can now be defined by the following vector equation.

$$\bar{e}_n = \frac{\bar{e}_t \times \bar{e}_c}{\bar{e}_t \times \bar{e}_c} \tag{2-32}$$

A fourth unit vector, $\bar{e}_r$, again contained in the plane of the curve, but directed towards the center of curvature, can now be defined as follows:

$$\bar{e}_r = \bar{e}_n \times \bar{e}_t \tag{2-33}$$

Designating the direction cosiness of $\bar{e}_r$ as α_r, β_r, and γ_r, the central included angle of the arc, θ, is given by the following equation. (Refer to Fig. 2.5-1).

$$\theta = 2 \arcsin (e_r \cdot e_c) \tag{2-34}$$

The radius of curvature, ρ, is given as follows:

$$\rho = \frac{L/2}{\bar{e}_r \cdot \bar{e}_c} \tag{2-35}$$

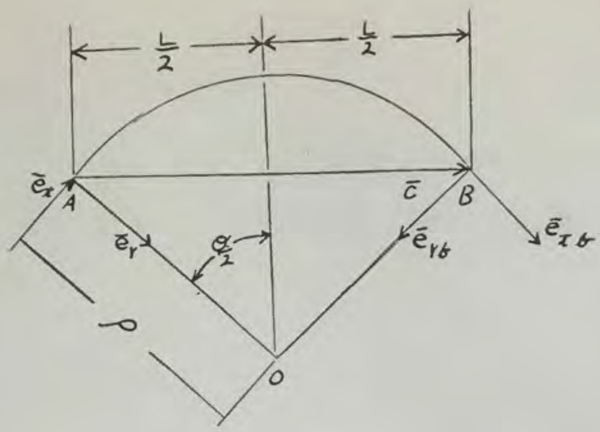


Fig. 2.5-1

The three unit vectors, $\bar{e}_x$, $\bar{e}_n$, and $\bar{e}_r$, form a local triad with origin at "a" which specifies the orientation of the state vector associated with the "a" end. A similar triad fulfilling the same function at the "b" end may be obtained as follows.

$$\overline{AO} = \overline{AB} + \overline{BO} \qquad 2\text{-}36$$

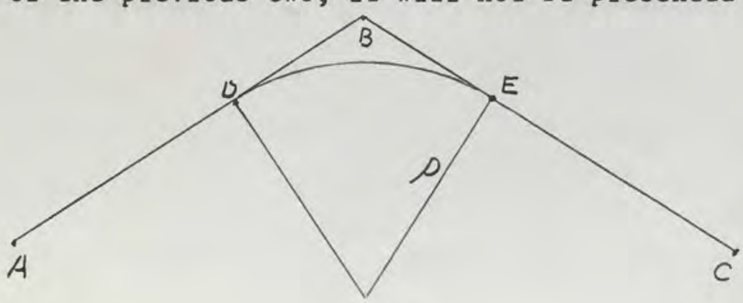
$$\rho\bar{e}_r = L\bar{e}_c + \rho\bar{e}_{rb} \qquad 2\text{-}37$$

$$\bar{e}_{rb} = \bar{e}_r - \frac{L\bar{e}}{\rho}_c \qquad 2\text{-}38$$

$$\bar{e}_{tb} = \bar{e}_{rb} \times \bar{e}_n \qquad 2\text{-}39$$

where $\bar{e}_{tb}$, $\bar{e}_n$, and $\bar{e}_{rb}$ forms the local triad at the "b" end.

A combination of two straight elements and a circular arc, such as Fig. 2.5-2, where one is given the system coordinates of three points, "a", "b", and "c", and the radius of curvature of an arc to be fitted in the included angle is also permitted. Since the required analysis is a combination of the previous two, it will not be presented here.



2.5a Node Rotation Matrix, [G].

It is now possible to discuss the geometry requirements associated with more than two triads located at one common node, but with different orientations.

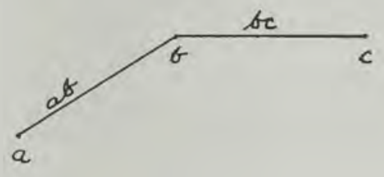


Fig. 2.5-3a

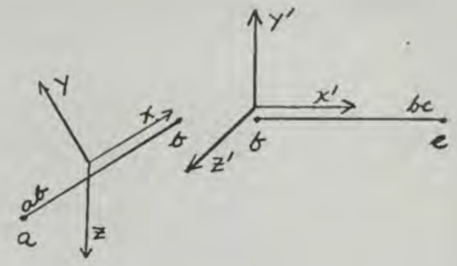


Fig. 2.5-3b

Consider two elements, ab and bc, connecting three nodes, a, b, and c, with node b a common node. If the centroidal axes of the two elements are not collinear at node b, as indicated in Fig. 2.5-3a, the triad associated with element ab and located at node b will not have the same orientation as the triad associated with element bc and also located at node b. In order to perform the multiplication indicated in eqs. 2-3, 2-4, and 2-6, it is necessary to perform a rotation transformation upon the state vector $[SV]_b$, as computed by eq. 2-2, before utilizing it in eq. 2-3 to compute $[SV]_c$. The circular arc transfer matrix developed in Appendix A performs the required rotation associated with a curved element as well as the translation associated with all transfer matrices.

Consider some element of $[S\hat{V}]_b$, as computed by eq. 2-2, oriented in the x direction of the triad associated with element ab at node b. It will have three components when expressed in the primed coordinates of the triad at node b associated with element bc. Since the triads concerned are defined by unit vectors expressed in system coordinates, the component of any given x directed element in the x' direction will be the product of the magnitude of the x directed element and the dot product, $\bar{e}'_x \cdot \bar{e}_x$.

As an example, let the first three deflection elements of $[SV]_b$ oriented with element ab have magnitudes sv1, sv2, and sv3. Let the desired magnitudes of the corresponding elements of $[SV]'_b$ oriented with element bc be designated sv1', sv2', and sv3'. The latter three values are computed by the following equations.

$$sv1' = \bar{e}'_x \cdot \bar{e}_x \cdot sv1 + \bar{e}'_x \cdot \bar{e}_y \cdot sv2 + \bar{e}'_x \cdot \bar{e}_z \cdot sv3 \qquad \text{2-40}$$

$$sv2' = \bar{e}'_y \cdot \bar{e}_x \cdot sv1 + \bar{e}'_y \cdot \bar{e}_y \cdot sv2 + \bar{e}'_y \cdot \bar{e}_z \cdot sv_3 \qquad \text{2-41}$$

$$sv3' = \bar{e}'_z \cdot \bar{e}_x \cdot sv1 + \bar{e}'_z \cdot \bar{e}_y \cdot sv2 + \bar{e}'_z \cdot \bar{e}_z \cdot sv3 \qquad \text{2-42}$$

or

$$\begin{bmatrix} sv1' \\ sv2' \\ sv3' \end{bmatrix} = \begin{bmatrix} \bar{e}'_x \cdot \bar{e}_x & \bar{e}'_x \cdot \bar{e}_y & \bar{e}'_x \cdot \bar{e}_z \\ \bar{e}'_y \cdot \bar{e}_x & \bar{e}'_y \cdot \bar{e}_y & \bar{e}'_y \cdot \bar{e}_z \\ \bar{e}'_z \cdot \bar{e}_x & \bar{e}'_z \cdot \bar{e}_y & \bar{e}'_z \cdot \bar{e}_z \end{bmatrix} \begin{bmatrix} sv1 \\ sv2 \\ sv3 \end{bmatrix} \qquad \text{2-43}$$

or

$$\begin{bmatrix} sv1' \\ sv2' \\ sv3' \end{bmatrix} = [GX] \cdot \begin{bmatrix} sv1 \\ sv2 \\ sv3 \end{bmatrix} \qquad \text{2-43a}$$

where

$$[GX] = \begin{bmatrix} e'_x \cdot e_x & e'_x \cdot e_y & e'_x \cdot e_z \\ e'_y \cdot e_x & e'_y \cdot e_y & e'_y \cdot e_z \\ e'_z \cdot e_x & e'_z \cdot e_y & e'_z \cdot e_z \end{bmatrix} \qquad \text{2-44}$$

Since the four sets of x, y, and z directed elements of a given state vector all require a similar transformation, it is apparent that a single 12 x 12 matrix will perform the desired transformation for all elements. Such a matrix will have the three by three matrix, [GX], repeated four times on the major diagonal as follows:

$$[G] = \begin{bmatrix} GX & 0 & 0 & 0 \\ 0 & GX & 0 & 0 \\ 0 & 0 & GX & 0 \\ 0 & 0 & 0 & GX \end{bmatrix} \qquad 2\text{-}45$$

The indicated elements of [G] are three by three matrices themselves. The bracket notation for [GX] has been omitted for clarity. This practice will be continued in the succeeding development.

Now the following complete transformation can be made

$$[SV]' = [G] \cdot [SV] \qquad 2\text{-}46$$

Equations 2-4 and 2-6 can now be written without omitting the necessary rotation matrix:

$$[SV]_c = [Z]_{bc} \cdot [G]_b \cdot [Z]_{ab} \cdot [SV]_a \qquad 2\text{-}4a$$

$$[Z]_{ac} = [Z]_{bc} \cdot [G]_b \cdot [Z]_{ab} \qquad 2\text{-}6a$$

2.6 Global Solid Geometry

In order to make use of certain features of linear graph theory which apply to mechanical network analysis, it is necessary to express all directed quantities in a single common reference system. Such a reference system will be called a global reference system. A discussion of network analysis of mechanical systems follows in Section 2.7.

The location and orientation of a global reference system may be chosen artibrarily. Although it could be anywhere, for the purposes of this discussion, the global reference system will have the same orientation and position as the first local triad to be constructed.

The derivation of the necessary transformation matrices and a more complete discussion are presented in Ref. 3. They are simply defined and exhibited here.

Let [F] be some force column matrix expressed in a local reference system and [F]' be the same force matrix expressed in the global reference system. [F] and [F]' will be related by a transformation matrix as follows:

$$[F]' = [TOA] \cdot [F] \qquad 2\text{-}47$$

The matrix [TOA] may be partitioned into four three by three matrixes as indicated below.

$$\left[\begin{array}{c|c} TOA_{11} & TOA_{12} \\ \hline TOA_{21} & TOA_{22} \end{array}\right]$$

TOA_{12} is a zero matrix.

TOA_{11} is obtained precisely like [GX] of Sect. 2.5 with the exception that the primed coordinate system is now the global reference system.

$$TOA_{22} = TOA_{11} \qquad 2\text{-}48$$

TOA_{21} is equal to TOA_{11} premultiplied by the following translation matrix [L].

$$L = \begin{bmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{bmatrix} \qquad 2\text{-}49$$

The elements of L are defined as follows.

Let X, Y, and Z be system coordinates of the local reference system position. Let X_g, Y_g, and Z_g be the system coordinates of the global reference system position. Then

$$L_1 = X - X_g \qquad 2\text{-}50$$

$$L_2 = Y - Y_g \qquad 2\text{-}51$$

$$L_3 = Z - Z_g \qquad 2\text{-}52$$

An inspection of [TOA] shows that the inverse is given as follows:

$$[TAO] = [TOA]^{-1} = \begin{bmatrix} TAO_{11} & 0 \\ \overline{TAO_{21}} & \overline{TAO_{22}} \end{bmatrix} \qquad 2\text{-}53$$

where

$$[TAO_{11}] = [TAO_{22}] = [TOA_{11}]^T \qquad 2\text{-}54$$

and

$$[TAO_{21}] = [TAO_{11}] \cdot [L]^T \qquad 2\text{-}55$$

The inverse transformation relationship can now be written as

$$[F] = [TAO] \cdot [F]' \qquad 2\text{-}56$$

A similar transformation exists for displacements.

Let [D] be some displacement matrix and let [D]' be the same displacement matrix expressed in a global reference system. The following equation may be written:

$$[D]' = [TAO]^T \cdot [D] \qquad 2\text{-}57$$

where

$$[TAO]^T = \begin{bmatrix} TOA_{11} & TOA_{21} \\ 0 & TOA_{22} \end{bmatrix} \qquad 2\text{-}58$$

The inverse relationship is also true.

$$[D] = [TOA]^T \cdot [D]' \qquad 2\text{-}59$$

where

$$[TOA]^T = \begin{bmatrix} TAO_{11} & TAO_{21} \\ 0 & TAO_{22} \end{bmatrix} \qquad 2\text{-}60$$

If the forces and deflections associated with a transfer matrix equation are transformed to a global form, a similar transformation must be made on the transfer matrix. A complete state vector is transformed as follows:

$$[SV]_b' = \begin{bmatrix} TAO_b^T & 0 \\ 0 & TOA_b \end{bmatrix} \cdot [SV]_b \qquad 2\text{-}61$$

or

$$[SV]_b' = [TLG]_b \cdot [SV]_b \qquad 2\text{-}61a$$

$$[SV]_a = \begin{bmatrix} TOA_a^T & 0 \\ 0 & TAO_a \end{bmatrix} \cdot [SV]'_a \qquad 2\text{-}62$$

$$\text{or} \quad [SV]_a = [TGL]_a \cdot [SV]'_a \qquad 2\text{-}62a$$

These transformations operate on the transfer matrix as follows:

$$[SV]_b = [Z]_{ab} \cdot [SV]_a \qquad 2\text{-}63$$

$$[SV]'_b = [TLG]_b \cdot [SV]_b = [TLG]_b \cdot [Z]_{ab} \cdot [TGL]_a \cdot [SV]'_a \qquad 2\text{-}64$$

or

$$[SV]'_b = [Z]'_{ab} \cdot [SV]'_a \qquad 2\text{-}64a$$

where

$$[Z]'_{ab} = [TLG]_b \cdot [Z]_{ab} \cdot [TGL]_a \qquad 2\text{-}65$$

A similar transformation exists for the dynamic stiffness matrix. The deflections are transformed as follows:

$$[D]_{a,b} = \begin{bmatrix} TOA_a^T & 0 \\ 0 & TOA_b \end{bmatrix} \cdot [D]'_{a,b} \qquad 2\text{-}66$$

$$\text{or} \quad [D]_{a,b} = [SGL]_{a,b} \cdot [D]_{a,b} \qquad 2\text{-}66a$$

$$[F]'_{a,b} = \begin{bmatrix} TOA_a & 0 \\ 0 & TOA_b \end{bmatrix} \cdot [F]_{a,b} \qquad 2\text{-}67$$

$$\text{or} \quad [F]'_{a,b} = [SLG]_{a,b} \cdot [F]_{a,b} \qquad 2\text{-}67a$$

$$[F]'_{a,b} = [SLG]_{a,b} \cdot [S]_{a,b} \cdot [SGL]_{a,b} \cdot [D]'_{a,b} \qquad 2\text{-}68$$

$$\text{or} \quad [F]'_{a,b} = [S]'_{a,b} \cdot [D]'_{a,b} \qquad 2\text{-}68a$$

where

$$[S]'_{a,b} = [SLG]_{a,b} \cdot [S]_{a,b} \cdot [SGL]_{a,b} \qquad 2\text{-}69$$

In the preceding development, one may think of the transformation symbols as follows:

[TLG], local to global transformation matrix for transfer matrices

[TGL], global to local transformation matrix for transfer matrices

[SLG], local to global transformation matrix for stiffness matrices

[SGL], global to local transformation matrix for stiffness matrices

2.7 Application of Graph Theory and Topological Matrices

The present treatment of the problem of vibration of a mechanical network with general topology utilizes certain features of the theory of Linear Graphs. This discussion will be limited to those features and will be developed in terms of mechanical parameters rather than abstract mathematical ones.

Given all the necessary quantities expressed in a global reference system, the topological solution which follows makes use of three conditions. First, the forces, applied by the primary path to the nodes upon which they are incident, are related to the end deflections of the paths by dynamic stiffness matrices. Second the nodes of the network must be in equilibrium, that is, the forces applied by the incident elements to a node must sum to zero for a primary node, or sum to the force which the node applies to the boundary in the case of a boundary node. Third, continuity requirements must be met at the nodes, that is, the deflections of the ends of the primary paths incident on a node are identically equal to each other and are equal to the deflection of the node.

Consider some network designated G. Let the number of nodes in G be M. Let the number of elements connecting the nodes be N/2. The N force column matrices associated with the N ends of the N/2 elements may be designated as $[FE_1]$, $[FE_2]$, $[FE_3]$,---,$[FE_N]$. The deflection matrices associated with the ends of the N/2 elements may be designated as $[DE_1]$, $[DE_2]$, $[DE_3]$,---, $[DE_N]$.

As defined by the stiffness matrices sign convention, the forces are positive when applied to the nodes. Now consider some node, m, of G, connected to the rest of G by n elements. In order that the node m be in equilibrium, the n forces associated with the elements incident on node m must sum to zero or to the force applied by the boundary to node m. The N-n forces that are not applied to node m do not enter into the summation. Let the boundary forces, which may be zero, be designated $[FN_m]$. There are M summations to be made, one for each node. A given summation may be indicated as follows:

$$[FN_m] = \sum_{j=1}^{N} [a^t]_{m,j} \cdot [FE_j] \qquad 2\text{-}70$$

The N coefficients $[a^t]_{m,j}$ are either a six by six identity matrix or a six by six zero matrix according to the following rule.

$$[a^t]_{m,j} = I \text{ if end } j \text{ is incident on node } m \qquad 2\text{-}71$$

$$[a^t]_{m,j} = 0 \text{ if end } j \text{ is not incident on node } m \qquad 2\text{-}71a$$

When the M summations have been made, the coefficients $[a^t]_{m,j}$ form a matrix designated $[A]^t$ which has 6M rows and 6N columns. Now, if the N force matrices are assembled in a single column matrix, [FE], and a second column matrix [FN], is formed which contains all the boundary forces, both zero and non-zero, then [FN] is related by equilibirum requirements to [FE] as follows:

$$[FN] = [A]^t \cdot [FE] \qquad 2\text{-}72$$

Next we concern ourselves with geometrical continuity at the same node m with n elements incident on it. The n deflections of the n element ends at m must be identical. Also, the deflection of the node itself must be identical to each of the n element deflections. Let the node deflection be $[DN_m]$. One of these identical relationships may be expressed as the following summation.

$$[DE_j] = \sum_{m=1}^{M} [k]_{j,m} \cdot [DN_m] \qquad 2\text{-}73$$

The M coefficients $[k]_{j,m}$ are either a six by six identity matrix or a six by six zero matrix according to the following rule.

$$[k]_{j,m} = [I] \text{ if end j is incident on node m} \qquad 2\text{-}74$$

$$[k]_{j,m} = [0] \text{ if end j is not incident on node m} \qquad 2\text{-}74a$$

There are N summations to be made. The coefficients $[k]_{j,m}$ form a matrix which is [K]. Inspection of the rules for forming [K] reveals that [K] is the transpose of $[A]^t$. Accordingly, it will be referred to as [A] henceforth.

If the N deflection matrices associated with element ends are assembled into one column matrix, [DE], and the M deflection matrices associated with nodes are assembled into one column matrix, [DN], then [DE] and [DN] are related by continuity requirements as follows:

$$[DE] = [A] \cdot [DN] \qquad 2\text{-}75$$

As developed in Section 2-4, eq. 2-12a, the forces and deflections associated with a given element, e, having ends a, b, are related as follows:

$$[F_e]_{a,b} = [S_e]_{a,b} \cdot [D_e]_{a,b} \qquad 2\text{-}76$$

There are N/2 such equations which can be written for G. Let them be written as follows:

$$[F_1]_{a,b} = [S_1]_{a,b} \cdot [D_1]_{a,b} \qquad 2\text{-}76.1$$

$$[F_2]_{a,b} = [S_2]_{a,b} \cdot [D_2]_{a,b} \qquad 2\text{-}76.2$$

$$[F_3]_{a,b} = [S_3]_{a,b} \cdot [D_3]_{a,b} \qquad 2\text{-}76.3$$

$$\vdots \quad = \quad \ddots \quad \vdots$$

$$[F_{N/2}]_{a,b} = [S_{N/2}]_{a,b} \cdot [D_{N/2}]_{a,b} \qquad 2\text{-}76.N/2$$

The N/2 equations may be written in more compact form as follows:

$$[FE] = [SP] \cdot [DE] \qquad 2\text{-}77$$

In eq. 2-77, the ordering of [FE] and [DE] follows a certain method. The forces occur in pairs for each element. This imposes a limitation on how the topological array [A] is made up. Although the order in which nodes occur is not restricted, it will be found most convenient to take the boundary nodes first and then the remainder in any order. The order in which the forces $[FE_n]$ have been numbered is governed by the manner in which the matrix [SP] is assembled.

If eq. 2-75 is substituted into eq. 2-77 for [DE], the following is obtained.

$$[FE] = [SP] \cdot [A] [DN] \qquad 2\text{-}78$$

If eq. 2-78 is premultiplied by $[A]^t$, and eq. 2-72 is used, the following is obtained.

$$[FE] = [A]^t \cdot [SP] \cdot [A] \cdot [DN] \qquad 2\text{-}79$$

which we will write as

$$[FN] = [SN] \cdot [DN] \qquad 2\text{-}79a$$

where

$$[SN] = [A]^t \cdot [SP] \cdot [A] \qquad 2\text{-}80$$

The matrix [SP] will be referred to as the primitive stiffness matrix. The matrix [SN] will be referred to as the node stiffness matrix.

2.8 Boundary Conditions and the Frequency Determinant.

A knowledge of the boundary constraints provides certain qualitative knowledge of [FN] and [DN] which in turn makes it possible to extract from [SN] an array, designated $[\overline{SN}]$, the determinant of which will vanish for certain values of frequency which are natural frequencies of the physical system.

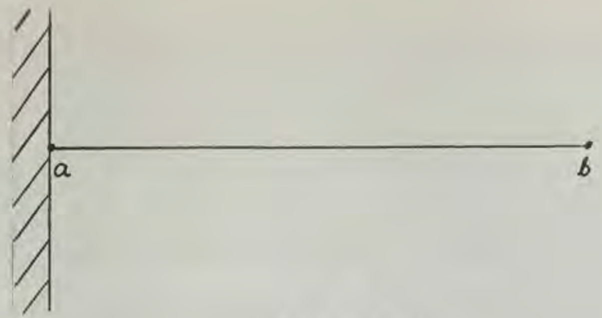


Fig. 2.8-1

If the problem consisted of the simple beam of Fig. 2.8-1, the two nodes a and b would both be boundary nodes. The deflections at b would be generally non-zero while the forces would be zero. At node a, the deflections would be zero while the forces would be non-zero. Depending on the boundary conditions at a given node, there may be any combination of zero and non-zero forces and deflections. However, there will always be the same number of zero quantities, that is, for each zero force there will be a non-zero deflection and vice versa. Reference 1 gives a more complete discussion of such boundary conditions.

The relationship [FN] = [SN] . [DN], as developed in Section 2.7, is a global equation while the given qualitative knowledge of [FN] and [DN] is in local coordinates. Just as forces and deflections were expressed in a global reference system so as to facilitate the desired topological operations, they must now be expressed in local coordinates so as to make use of the given qualitative knowledge of boundary conditions. However, only those six element matrices containing both zero and non-zero elements in local coordinates need be expressed in local coordinates. If [FN] and [DN] have not been assembled so that those six element matrices just mentioned are adjacent and in low numbered positions, let [FN], [DN], and [SN] be reordered as necessary so that the qualitative information is so positioned. If the number of nodes is M and the number of those having non-zero $[FN_m]$ matrices is p, then [FN] may be partitioned into two column matrices, the first containing p six element matrices and the second containing (m-p) six element matrices, of which, all elements are zero in global or in local coordinates. [DN] may be partitioned in like manner. The given qualitative knowledge of the first p six element matrices is not available in global coordinates, but the second will contain m-p six element matrices, which will be non-zero in either global or local coordinates.

Let [FN], [DN], and [SN] be partitioned in this manner. Then the node stiffness equation* may be written as follows:

$$\begin{bmatrix} F_p \\ \hline 0 \end{bmatrix} = \left[\begin{array}{c|c} sn_{11} & sn_{12} \\ \hline sn_{21} & sn_{22} \end{array}\right] \cdot \begin{bmatrix} D_p \\ \hline D_{m-p} \end{bmatrix} \qquad 2\text{-}81$$

$$[F_p] = [sn_{11}] \cdot [D_p] + [sn_{12}] \cdot [D_{m-p}] \qquad 2\text{-}81a$$

$$[0] = [sn_{21}] \cdot [D_p] + [sn_{22}] \cdot [D_{m-p}] \qquad 2\text{-}81b$$

The forces and deflections associated with the p boundary nodes have the same local position and orientation as the forces and deflections associated with the element end incident on a given boundary node. Therefore, the inverse of the p translation and rotation matrices, [TOA] and $[TAO]^t$, which transformed the boundary node element end forces and deflections to a global reference system may be retrieved and utilized to transform $[F_p]$ and $[D_p]$ to local coordinates. Let this be done and let the appropriate transformation matrices be arranged in two large arrays as indicated below:

$$\begin{bmatrix} TAO_1 & 0 & 0 \\ 0 & TAO_2 & 0 \\ & & \\ 0 & 0 & TAO_p \end{bmatrix} = [TAOP] \qquad 2\text{-}82$$

*cf. Eq. 2-79a.

$$\begin{bmatrix} TOA_1^t & 0 & 0 \\ 0 & TOA_2^t & 0 \\ & & \\ 0 & 0 & TOA_p^t \end{bmatrix} = [TOAP]^t \qquad 2\text{-}83$$

Then we can write

$$[F_p]_{local} = [TAOP] \cdot [F_p] \qquad 2\text{-}84$$

and

$$[D_p] = [TOAP]^t \cdot [D_p]_{local} \qquad 2\text{-}85$$

Premultiplying eq. 2-81a by [TAOP] gives the following result.

$$[TAOP] \cdot [F_p] = [F_p]_{local} = [TAOP] \cdot [sn_{11}] \cdot [D_p] +$$

$$[TAOP] \cdot [sn_{12}] \cdot [D_{m-p}] \qquad 2\text{-}86$$

Substituting eq. 2-85 into eq. 2-81b and eq. 2-86 for $[D_p]$ gives the following two equations.

$$[F_p]_{local} = [TAOP] \cdot [sn_{11}] \cdot [TOAP]^t \cdot [D_p]_{local} +$$

$$[TAOP] \cdot [sn_{12}] \cdot [D_{m-p}] \qquad 2\text{-}87$$

$$[0] = [sn_{21}] \cdot [TOAP]^t \cdot [D_p]_{local} + [sn_{22}] \cdot [D_{m-p}] \qquad 2\text{-}88$$

Let the following four arrays be defined as indicated.

$$[sn_{11}]' = [TAOP] \cdot [sn_{11}] \cdot [TOAP]^t \qquad 2\text{-}89$$

$$[sn_{12}]' = [TAOP] \cdot [sn_{12}] \qquad 2\text{-}90$$

$$[sn_{21}]' = [sn_{21}2 \cdot [TOAP]^t \qquad 2\text{-}91$$

$$[sn_{22}]' = [sn_{22}] \qquad 2\text{-}92$$

Now equations 2-81a and 2-81b may be written as follows.

$$\begin{bmatrix} F_p \\ \hline 0 \end{bmatrix} = [SN]' \cdot \begin{bmatrix} D_p \\ \hline D_{m-p} \end{bmatrix} \qquad 2\text{-}93$$

where $[F_p]$ and $[D_p]$ are now in local coordinates.

There are 6p elements in $[F_p]$ and also in $[D_p]$. Just as [SN] was reordered so that boundary node forces and deflections were in low numbered positions in [FN] and [DN], [SN]' may be reordered so that the non-zero elements of $[F_p]$ are in low numbered positions and the zero elements of $[D_p]$ are in corresponding positions. For m nodes, [SN]' would be a 6m by 6m matrix. If the number of non-zero elements of $[F_p]$ were q, then [SN]' could be partitioned as follows:

$$[F_q] = [sn_{11}]'' \cdot [0] + [sn_{12}]'' \cdot [D_{m-q}] \qquad 2\text{-}94$$

$$[0] = [sn_{21}]'' \cdot [0] + [sn_{22}]'' \cdot [D_{m-q}] \qquad 2\text{-}95$$

For the boundary conditions given above, $[sn_{11}]''$ is q by q, $[sn_{12}]''$ is q by m-q, $[sn_{21}]''$ is m-q by q, and $[sn_{22}]''$ is m-q by m-q. Inspection of $[sn_{22}]''$ reveals that it must be singular if the non-zero elements of [DN] may have any given value. The determinant of $[sn_{22}]''$ will be referred to as the frequency determinant, $|\overline{SN}|$. By evaluating $|\overline{SN}|$ at various frequencies over a given range, those values of frequency which yield a zero value for $|\overline{SN}|$ that is the eigenvalues or characteristic frequencies, may be determined.

The reordering procedure indicated in the preceding is most appropriate for computer procedures. If one were doing such a manipulation with pencil and paper, one would naturally strike out rows of [SN]' corresponding to non-zero elements of $[F_p]$ and strike out columns of [SN]' corresponding to zero elements of $[D_p]$. The resulting array would be $[sn_{22}]''$. We may now adopt a more convenient notation for $[sn_{22}]''$, that is $[\overline{SN}]$.

2.9 Mode Shapes

If we presume that $[\overline{SN}]$ is evaluated at a characteristic frequency and that $[\overline{SN}]$ is only "singly singular", that is, its rank is one less than its order*, we have

$$0 = [\overline{SN}] \cdot [DM] \qquad 2\text{-}96$$

Let [DM] have g elements. One, say g_a, can always be chosen such that the other g-1 elements of [DM] can be expressed in terms of g_a. [DM] can then be normalized so that

$$[DM]^t \cdot [DM] = 1 \qquad 2\text{-}97$$

The matrix, [DM], thus obtained is unique, that is, independent of our choice of g_a.

[DM] is the mode shape of the primary nodes. Trivial nodes of interest may be included in [DM] by treating them as primary nodes in all the preceding analysis.

An optional method of obtaining the mode shape of the trivial nodes is as follows:

Reconstruct [DN] including the elements of [DM] where appropriate and carrying out the necessary local to global transformations. Obtain [DE] by pre-multiplying [DN] by [A] as follows:

$$[DE] = [A] \cdot [DN] \qquad 2\text{-}98$$

The matrix [DE] may then be partitioned into the twelve element column matrices associated with the primary paths.

*Otherwise, we have the coalescence of two roots which would complicate subsequent discussion unnecessarily, the situation being essentially no different than the case of any vibrating system having identical characteristic frequencies, corresponding to which the modes are not unique.

For any given primary path, say between primary nodes a and b, we may then write

$$\begin{bmatrix} F \\ - \\ F \end{bmatrix}\begin{matrix} a \\ \\ b \end{matrix} = [S]_{a,b} \cdot \begin{bmatrix} D \\ - \\ D \end{bmatrix}\begin{matrix} a \\ \\ b \end{matrix} \qquad 2\text{-}99$$

Solving eq. 2-99 for $[F]_a$ and $[F]_b$ provides sufficient knowlėdge to assemble a complete state vector at either node a or node b. By choosing node b, we may avoid any confusion concerning sign conventions. We may then write $[SV]_b$ as follows:

$$[SV]_b = \begin{bmatrix} D \\ - \\ f \end{bmatrix}\begin{matrix} b \\ b \end{matrix} \qquad 2\text{-}100$$

In eq. 2-100, we use lower case f indicating forces and sign conventions appropriate for transfer matrix use. We may then evaluate the transfer matrices, at the appropriate eigenvalues, between the primary nodes and proceed to obtain the deflections associated with all nodes.

CHAPTER III

Discussion

3.1 Network Analogies

As has been previously stated, the solution for characteristic frequencies of three dimensional piping systems with general topology requires a form of network analysis. Once all vector like quantities had been expressed in a single common global reference system, as suggested by Ref. 3, the writer found the methods of network analysis developed by Fenves and Branin, Seshu and Reed, and Kron to be applicable to the subject analysis.

Kron, in Ref. 5, describes the equivalent network of a vibrating beam as a six wire, six phase transmission line. Such a model does not permit the direct application of the development of Ref. 3. However, a similar technique was found in Ref. 4 which was directly applicable to a mechanical network when the properties of such a network were described in terms of the dynamic stiffness matrix suggested by Pestel in Ref. 1.*

It should be noted that the mechanical transfer matrix was found to be analogous to the electrical transfer matrix of h parameters. The dynamic stiffness matrix was found to be analogous to the electrical short circuit admittance matrix. The connection tensors of Ref. 4 are node incidence matrices. And finally, force and current, as well as voltage and deflection, are analogous.

3.2 Practical Limitations

In principle, there are no limitations to the size of a mechanical system that can be treated in this manner.

*cf. page 150, para. 3 of Ref. 1

In reality, there are two. First, any given digital computer, though possessing total recall, has a limited memory capacity. The method developed in this thesis requires the assembly of very large arrays. A network composed of 20 non-trivial nodes and 30 primary paths requires storage capacity for 57,600 quantities for just two of the matrices involved, no matter how clever the programmer.* Although the subsystem technique developed in App. B would relieve this requirement somewhat, very large high speed computers are still mandatory computational tools. The second limitation is that of the size of the numbers involved. A very large system will require the ability to compute very large or very small numbers. Thus, double, or even triple, precision may be required in order that precise answers may be obtained.

*For the example above, $[A]^t \cdot [SP]$ would be dimensioned 120 by 360 and would contain 43,200 elements. $[A]^t \cdot [SP] \cdot [A]$ would be dimensioned 120 by 120 and would contain 14,400 elements.

BIBLIOGRAPHY

1. Pestel, E. C., Matrix Methods in Elastomechanics. McGraw-Hill Book Company, New York, 1963.

2. Fink, George E., Vibration Analysis of Piping Systems. Masters Degree Thesis, United States Naval Postgraduate School, 1964.

3. Fenves, S. J. and Branin, F. H., Jr., A Network-Topological Formulation of Structural Analysis. Technical Report #00.979, Feb. 25, 1963. IBM Data Systems Division Development Laboratory, Poughkeepsie, New York.

4. Kron, G., Tensor Analysis of Networks. John Wiley and Sons, Inc., New York, 1939.

5. Kron, G., Tensor Analysis and Equivalent Circuits of Elastic Structures. Journal of the Franklin Institute, Dec., 1944. Vol. 238, No. 6, pp. 400.

6. Kron, G., Solving Highly Complex Elastic Structures in Easy Stages. Journal of Applied Mechanics, June, 1955. Vol. 22, No. 2, pp. 235.

7. Seshu, S., and Reed, Myril B., Linear Graphs and Electrical Networks. Addison-Wesley Publishing Company, Inc., Reading, Mass., 1961.

APPENDIX A

A. Distributed Mass Transfer Matrix for Circular Arcs

A.1 General Remarks

Although it is not the purpose of this thesis to derive transfer matrixes, the fact that the transfer matrix for a distributed mass model of a circular arc has not been developed and utilized before, warrants a discussion of the technique employed. Ref. 1 indicates the general method.*

Since the development which follows is restricted to a single planar element, it is both possible and convenient to consider an In-Plane case and an Out-of-Plane case. Consider the In-Plane case. A satisfactory state vector, consisting of three deflections and three forces, may be defined as follows using the notation of Sect. 2.2.

$$[SV]_{IP} = \begin{bmatrix} sv1 \\ sv2 \\ sv6 \\ sv12 \\ sv7 \\ sv8 \end{bmatrix} \qquad \text{A-1}$$

Given such a state vector, the following equation, as suggested by Ref. 1, may be written:

$$\frac{d[SV]_{IP}}{ds} = [A] \cdot [SV]_{IP} \qquad \text{A-2}$$

where s indicates the length of the arc. The matrix [A] should not be confused with the topological matrices previously developed.

*cf. page 145, para. 3 of Ref. 1.

Matrix, [A], is an array of coefficients of six simultaneous differential equations. It is derived in Ref. 1 and is repeated below:

$$
[A] = \begin{bmatrix}
0 & -K & 0 & 0 & 0 & \frac{1}{EA} \\
K & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{EI_Z} & 0 & 0 \\
0 & 0 & ui_z^2 w^2 & 0 & -1 & 0 \\
0 & -uw^2 & 0 & 0 & 0 & K \\
-uw^2 & 0 & 0 & 0 & -K & 0
\end{bmatrix} \qquad \text{A-3}
$$

The parameters of the non-zero elements of [A] are defined as follows:

K , curvature, or inverse of radius

E, Young's modulus

I_z , moment of inertia about the z axis

u, mass per unit length

i_z , radius of gyration of cross sectional area about the z axis

w, circular frequency of free vibration

The elements of [A], for convenience, are renamed as follows after multiplication by s_0 , total length of arc.

$$Ks_o = B$$

$$\frac{s_o}{EA} = C$$

$$\frac{s_o}{EI} = H$$

$$ui_z^2 w\ s_o = J$$

$$-uw^2 s_o = L$$

$[A]s_o$ may be reordered in the following steps:

(a) interchange rows 2 and 5

(b) interchange columns 2 and 5

The following is thus obtained:

$$[A]_{s_o} = \begin{bmatrix} 0 & 0 & 0 & 0 & -B & C \\ 0 & 0 & 0 & 0 & L & B \\ 0 & 0 & 0 & H & 0 & 0 \\ 0 & -s_o & J & 0 & 0 & 0 \\ B & 0 & s_o & 0 & 0 & 0 \\ L & -B & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{A-4}$$

As indicated by Ref. 1, the eigenvalues, λ , of the following array must now be obtained.

$$\Big[[I]\lambda - [A]\ s_0 \Big] = [SD]$$

It is apparent that [SD] has the following form:

$$[SD] = \left[\begin{array}{ccc|ccc} \lambda & 0 & 0 & 0 & +B & -C \\ 0 & \lambda & 0 & 0 & -L & -B \\ 0 & 0 & \lambda & -H & 0 & 0 \\ \hline 0 & +s_o & -J & \lambda & 0 & 0 \\ -B & 0 & -s_o & 0 & \lambda & 0 \\ -L & +B & 0 & 0 & 0 & \lambda \end{array}\right] \qquad \text{A-5}$$

The characteristic equation of [SD] may be obtained in a number of ways. The following matrix reduction scheme is most convenient. Let [SD] be partitioned into four three by three matrices as indicated above and below.

$$[SD] = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] \qquad \text{A-6}$$

A reduced matrix, [SDR], having the same characteristic values, is given as follows:

$$[SDR] = [P] - [Q] \cdot [S]^{-1} \cdot [R] \qquad \text{A-7}$$

$$[SDR]\lambda = \lambda^2 \, [I] - [Q] \cdot [R]$$

The product, [Q] . [R], is given below:

$$[Q] \cdot [R] = \begin{bmatrix} (-B^2 + CL) & (-CB) & (-B)s_o \\ (2LB) & (-B^2) & (L)s_o \\ 0 & (-H)s_o & (HJ) \end{bmatrix} \qquad \text{A-8}$$

Now, the elements of $[SDR]\lambda$ have the following form.

$$[SDR]\lambda = \begin{bmatrix} (\lambda^2 - CL + B^2) & (CB) & (B)s_o \\ (-2LB) & (\lambda^2 + B^2) & (-L)s_o \\ 0 & (H)s_o & (\lambda^2 - HJ) \end{bmatrix} \qquad \text{A-9}$$

Expansion by cofactors reveals the following polynominal as the characteristic equation, or C. E.

$$\text{C.E.} = (\lambda^2 + B^2 - CL)(\lambda^2 + B^2)(\lambda^2 - HJ) + (\lambda^2 + B^2 - CL)(LH)s_o^2 + (2LB)(CB)(\lambda^2 - HJ) - (2LB)(H)(B)s_o^2 \qquad \text{A-10}$$

Further expansion reveals eq. A-10 as the following cubic in λ^2.

$$\text{C.E.} = (\lambda^2)^3 + (2B^2 - HJ - CL)(\lambda^2)^2 + (B^2 - 2HJB^2 + HJCL + CLB^2 + LHs^2)(\lambda^2) - (LHB^2s^2 + HJCLB^2 + CL^2Hs^2 + HJB^4) \qquad \text{A-10a}$$

Further algebraic manipulation of the characteristic equation would be very tedious and difficult. However, numerical solution is quite simple. The three coefficients of the powers of λ^2 may be evaluated for an assumed value of frequency. The trigonometric solution of a cubic equation will yield three values for λ^2. The square roots of these values may then be obtained yielding six values of λ , generally complex.

The solution of eq. A-2 is of the following form.

$$[SV]_b = e^{[A]s_o} \cdot [SV]_a \qquad \text{A-11}$$

or

$$[SV]_b = [Z]_{a,b} \cdot [SV]_a \qquad \text{A-11a}$$

where

$$[Z]_{a,b} = e^{[A]s_o} \qquad \text{A-12}$$

Now the Cayley-Hamilton theorem may be employed as suggested in Ref. 1. Since $[A]s_o$ is a sixth-order square matrix, the (in-plane) transfer matrix may be written as follows:

$$[Z]_{IP} = e^{[A]s_o} = c_o[I] + c_1[A]s_o + c_2([A]s_o)^2 + c_3([A]s_o)^3 +$$

$$c_4([A]s_o)^4 + c_5([A]s_o)^5 \qquad \text{A-13}$$

Since the eigenvalues of $[A]s_o$ must satisfy eq. A-13 also, six simultaneous equations may be written in a form similar to eq. A-13. These six equations may be written in matrix form as indicated below:

$$\begin{bmatrix} e^{\lambda_1} \\ e^{\lambda_2} \\ e^{\lambda_3} \\ e^{\lambda_4} \\ e^{\lambda_5} \\ e^{\lambda_6} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 & \lambda_1^5 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 & \lambda_2^5 \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 & \lambda_3^4 & \lambda_3^5 \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \lambda_4^4 & \lambda_4^5 \\ 1 & \lambda_5 & \lambda_5^2 & \lambda_5^3 & \lambda_5^4 & \lambda_5^5 \\ 1 & \lambda_6 & \lambda_6^2 & \lambda_6^3 & \lambda_6^4 & \lambda_6^5 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} \qquad \text{A-14}$$

or

$$[ECS] = [RI\lambda].[C] \qquad \text{A-14a}$$

A given value of λ may be represented as a complex number, $R(\lambda) + jI(\lambda)$, where $R(\lambda)$ indicates the real part of λ and $I(\lambda)$ indicates the imaginary part of λ. A power of λ may be indicated in a similar fashion as $R(\lambda^n) + jI(\lambda^n)$. When a value of λ has a non-zero imaginary part, there will be another value of λ which will be the complex conjugate. When $I(\lambda)$ is zero, the corresponding rows of [ECS] and [RIλ] will remain as shown above.

If $I(\lambda)$ is non-zero, then the two rows of [ECS] and [RIλ] corresponding to the conjugate pair involved may be combined as follows to yield equations containing real numbers only.

$$e^{R(\lambda)}\cos I(\lambda) = c_0 + c_1R(\lambda^2) = c_3R(\lambda^3) + c_4R(\lambda^4) + c_5R(\lambda^5) \qquad \text{A-15}$$

$$e^{R(\lambda)}\sin I(\lambda) = c_0 + c_1I(\lambda) + c_2I(\lambda^2) + c_3I(\lambda^3) + c_4I(\lambda^4) + c_5I(\lambda^5) \qquad \text{A-15a}$$

These two equations may be substituted in place of the rows from which they were obtained so as to obtain real values only in eq. A-14 and eq. A-14a. Then the coefficients may be found as

$$[C] = [RI\lambda]^{-1}.\ [ECS] \qquad \text{A-16}$$

Having thus obtained the six coefficients of eq. A-13, $[Z]_{IP}$ may readily be evaluated for the assumed frequency. A similar procedure will yield the Out-of-Plane transfer matrix $[Z]_{OP}$. $[Z]_{IP}$ and $[Z]_{OP}$ may then be combined into one twelve by twelve array which, when reordered, is the general three dimensional transfer matrix desired.

APPENDIX B

B. Sub-system Topological Technique

Extremely large piping systems would exceed the storage capacity of the large computers available. A sub-system technique is presented in this appendix, which may ease computer storage problems.

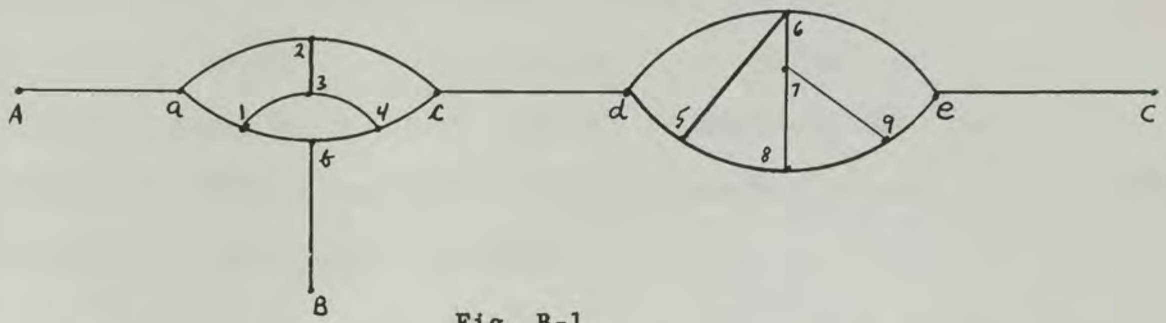


Fig. B-1

The system represented in Fig. B-1 might be viewed as the four elements connecting nodes A and a, B and b, C and c, c and d plus two sub-systems, one connecting nodes a, b, c, and containing nodes 1, 2, 3, and 4, and a second sub-system connecting nodes d and e and containing nodes 5, 6, 7, 8, and 9.

Let the first sub-system be considered isolated from the general system at nodes a, b, and c, as represented in Fig. B-2.

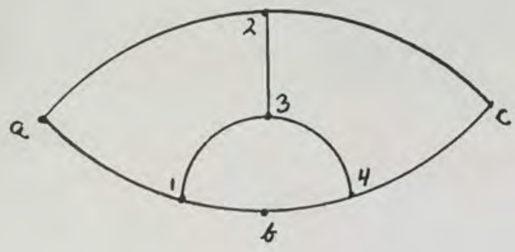


Fig. B-2

The node stiffness matrix, [SN], may be obtained in the fashion previously described. This matrix would be 42 x 42. The node force matrix,

[FN], would have, when appropriately ordered, 18 non-zero elements in the low numbered positions and, since there are no forces applied to the subsystem at nodes other than a, b, and c; the remaining 24 elements of [FN] would be zero. All of the elements of the node displacement matrix [DN] would be generally non-zero. Let [FN] be partitioned into a non-zero matrix, $[FN]_1$, corresponding to the forces at nodes a, b, and c, and a zero matrix $[FN]_2$. Let [DN], ordered so that the first 18 elements correspond to the 18 elements of $[FN]_1$, be partitioned so that the first 18 elements make up a matrix, $[DN]_1$, and the remaining elements make up a second matrix, $[DN]_2$. Let [SN] be reordered and partitioned so that the following two equations may be written:

$$[FN]_1 = [R] \cdot [DN]_1 + [S] \cdot [DN]_2 \qquad \text{B-1}$$

$$[FN]_2 = [T] \cdot [DN]_1 + [U] \cdot [DN]_2 \qquad \text{B-2}$$

where

$$[SN] = \begin{bmatrix} R & S \\ - & - \\ T & U \end{bmatrix} \qquad \text{B-3}$$

Since $[FN]_2$ is a zero matrix, eq. B-2 may be used to define $[DN]_2$ as a function of $[DN]_1$, [T], and [U] as follows:

$$[0] = [T] \cdot [DN]_1 + [U] \cdot [DN]_2 \qquad \text{B-2a}$$

Premultiply by the inverse of [U].

$$-[U]^{-1} \cdot [T] \cdot [DN]_1 = [DN]_2 \qquad \text{B-4}$$

substitute eq. B-4 into eq. B-1 for $[DN]_2$.

$$[FN]_1 = [R] \cdot [DN]_1 + [S] \cdot (-[U]^{-1} \cdot [T] \cdot [DN]_1) \qquad \text{B-5}$$

or

$$[FN]_1 = ([R] - [S] \cdot [U]^{-1} \cdot [T]) \cdot [DN_1] \qquad \text{B-5a}$$

or

$$[FN]_1 = [SN]_1 \cdot [DN_1] \qquad \text{B-5b}$$

The column matrix $[FN]_1$ contains the forces applied by the sub-system to nodes a, b, and c. Let it be written as follows:

$$[FN]_1 = \begin{bmatrix} FE_{1a} \\ \hline FE_{1b} \\ \hline FE_{1c} \end{bmatrix} \qquad \text{B-6}$$

In a similar manner, write $[DN]_1$ as follows:

$$[DN]_1 = \begin{bmatrix} DE_{1a} \\ \hline DE_{1b} \\ \hline DE_{1c} \end{bmatrix} \qquad \text{B-7}$$

Now, eq. B-5b may be written as follows:

$$\begin{bmatrix} FE_{1a} \\ \hline FE_{1b} \\ \hline FE_{1c} \end{bmatrix} = [SN]_1 \cdot \begin{bmatrix} DE_{1a} \\ \hline DE_{1b} \\ \hline DE_{1c} \end{bmatrix} \qquad \text{B-5c}$$

If a similar analysis were made for the second sub-system, the following would result.

$$\begin{bmatrix} FE_{2d} \\ \hline FE_{2e} \end{bmatrix} = [SN]_2 \quad \begin{bmatrix} DE_{2d} \\ \hline DE_{2e} \end{bmatrix} \qquad \text{B-8}$$

Similar equations can be written for those elements connecting the sub-systems together and to the boundaries at A, B, and C.

The system of Fig. B-1 has been thus reduced to that of Fig. B-3.

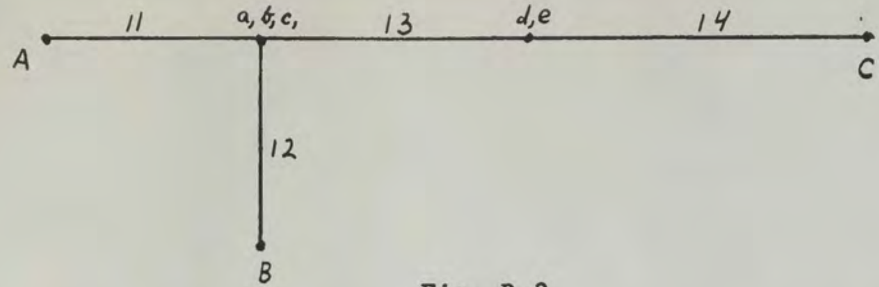


Fig. B-3

The reduced node incidence matrix [A] may be written as follows:

$$
\begin{array}{c}
\text{element} \\
\text{ends} \\
11_A \\ 11_a \\ 12_B \\ 12_b \\ 13_c \\ 13_d \\ 14_c \\ 14_e \\ 1_a \\ 1_b \\ 1_c \\ 2_d \\ 2_e
\end{array}
\begin{array}{c}
\text{Nodes} \\
\begin{array}{cccccccc} A & B & C & a & b & c & d & e \end{array} \\
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

The primitive stiffness matrix of the reduced system has the following form:

$$[SP] = \begin{bmatrix} S_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & S_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & S_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & S_2 \end{bmatrix}$$

S_{11}, S_{12}, S_{13}, S_{14}, and S_2 are 12 x 12 arrays in this particular problem while S_1 is an 18 x 18 array.

The reduced system node stiffness matrix may now be formed from the reduced node incidence matrix and the reduced primitive stiffness matrix.

$$[SN] = [A]^T \cdot [SP] \cdot [A] \qquad \text{B-9}$$

The reduced system frequency determinant, $|\overline{SN}|$, the characteristic frequencies and the characteristic mode shapes may now be obtained in the fashion previously developed.

In the preceding developments, trivial nodes were eliminated by using transfer matrix techniques. In this Appendix, we have seen that there are other nodes, which might be referred to as "removable nodes", which may be eliminated by the indicated manipulation.